

## SCALAR GENERALIZED VERMA MODULES

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ABSTRACT. In this paper we study the Verma module  $M(\mu)$  associated to a linear form  $\mu \in \mathfrak{h}^*$  where  $\mathfrak{sl}(E) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$  is a triangular decomposition of  $\mathfrak{sl}(E)$ . The  $\mathrm{SL}(E)$ -module  $M(\mu)$  has a canonical simple quotient  $L(\mu)$  with a canonical generator  $v$ . We study the left annihilator ideal  $\mathrm{ann}(v)$  in  $U(\mathfrak{sl}(E))$ . We also study scalar generalized Verma module  $M(\rho)$  associated to a character  $\rho$  of  $\mathfrak{p}$  where  $\mathfrak{p}$  is a parabolic subalgebra of  $\mathfrak{sl}(E)$ . We prove  $M(\rho)$  has a canonical simple quotient  $L(\rho)$ . This simple quotient is in some cases an infinite dimensional  $\mathrm{SL}(E)$ -module. We get a class of mutually non-isomorphic irreducible  $\mathrm{SL}(E)$ -modules containing the class of all finite dimensional irreducible  $\mathrm{SL}(E)$ -modules. As a result we give an algebraic proof of a classical result of Smoke on the structure of the jet bundle as  $P$ -module on any flag-scheme  $\mathrm{SL}(E)/P$  where  $P$  is any parabolic subgroup.

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## 1. INTRODUCTION

We study the scalar generalized Verma module  $M(\rho)$  associated to a character  $\rho$  of  $\mathfrak{p}$  where  $\mathfrak{p}$  is a parabolic subalgebra of  $\mathfrak{sl}(E)$ . We prove  $M(\rho)$  has a canonical simple quotient  $L(\rho)$ . This simple quotient is in some cases an infinite dimensional  $\mathrm{SL}(E)$ -module. We get a construction of a class of mutually non-isomorphic irreducible  $\mathrm{SL}(E)$ -modules containing the class of finite dimensional irreducible  $\mathrm{SL}(E)$ -modules. This class contains many infinite dimensional  $\mathrm{SL}(E)$ -modules.

Let  $V_\lambda$  be an arbitrary finite dimensional irreducible  $\mathrm{SL}(E)$ -module. In this note we give a construction of  $V_\lambda$  using the enveloping algebra  $U(\mathfrak{sl}(E))$  and the Verma modules  $M(\mu)$ . We study the annihilator ideal  $\mathrm{ann}(v)$  in  $U(\mathfrak{sl}(E))$  where  $v$  is the highest weight vector in  $V_\lambda$ . We prove the class of simple modules  $L(\rho)$  may be constructed using classical Verma modules  $M(\mu)$  as done by Dixmier in [1]. As a

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result we give an algebraic proof of a classical result of Smoke on the structure of the jet bundle as  $P$ -module on  $\mathrm{SL}(E)/P$  (see Corollary 3.19).

## 2. SCALAR GENERALIZED VERMA MODULES

In this section we construct any scalar generalized Verma module  $M(\rho)$ . Here  $\rho$  is a character of  $\mathfrak{p}$  where  $\mathfrak{p}$  is any parabolic subalgebra of  $\mathfrak{sl}(E)$ . We prove  $M(\rho)$  has a canonical simple quotient  $L(\rho)$ . When  $L(\rho)$  is finite dimensional we get a construction of all finite dimensional irreducible  $\mathrm{SL}(E)$ -modules.

Let  $G = \mathrm{SL}(E)$  where  $E$  is an  $n$ -dimensional vector space over an algebraically closed field  $K$  of characteristic zero and let  $\mathfrak{g} = \mathfrak{sl}(E)$ . Let  $\mathfrak{h}$  be the abelian subalgebra of  $\mathfrak{sl}(E)$  of diagonal matrices. It follows  $(\mathfrak{sl}(E), \mathfrak{h})$  is a split semi simple Lie algebra and determines a root system  $R = R(\mathfrak{sl}(E), \mathfrak{h})$ . Let  $B$  be a basis for  $R$ . This determines the positive roots  $R_+$  and the negative roots  $R_-$ . This determines a triangular decomposition  $\mathfrak{sl}(E) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$  of  $\mathfrak{sl}(E)$ .

Let  $1 \leq n_1 < n_2 < \dots < n_k \leq n-1$  be integers where  $\dim(E) = n$  and let  $l_1, \dots, l_k \in \mathbb{Z}$ . Let  $d_1 = n_1$ ,  $d_i = n_i - n_{i-1}$  for  $i = 2, \dots, n-1$  and  $d_n = n - n_k$ . Let  $\underline{l} = (l_1, \dots, l_k)$  and  $\underline{n} = (n_1, \dots, n_k)$ . Let  $e_1, \dots, e_n$  be a basis for  $E$  as  $K$ -vector space and let  $E_i = K\{e_1, \dots, e_{n_i}\}$ . Let  $P$  in  $\mathrm{SL}(E)$  be the subgroup fixing the flag

$$E_\bullet : 0 \neq E_1 \subseteq \dots \subseteq E_k \subseteq E$$

in  $E$ . Let  $\mathfrak{p} = \mathrm{Lie}(P)$ . It follows  $\mathfrak{p}$  consists of matrices on the form

$$x = \begin{pmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \dots & * \\ 0 & \vdots & \dots & * \\ 0 & 0 & \dots & A_{k+1} \end{pmatrix}$$

where  $A_i$  is a  $d_i \times d_i$ -matrix and  $\mathrm{tr}(x) = 0$ . Let  $L_\rho = Kw$  be a rank one  $K$ -vector space on the element  $w$ . Define the following character  $\rho_{\underline{l}} = \rho$

$$\rho : \mathfrak{p} \rightarrow \mathrm{End}(L_\rho)$$

by

$$\rho(x) = \eta(x)w = \sum_{i=1}^k l_i (\mathrm{tr}(A_1) + \dots + \mathrm{tr}(A_i))w.$$

**Lemma 2.1.** *The pair  $(L_\rho, \rho)$  is a rank one  $\mathfrak{p}$ -module. All rank one  $\mathfrak{p}$ -modules arise in this way.*

*Proof.* The proof is an exercise. □

Let  $U_l(\mathfrak{sl}(E)) \subseteq U(\mathfrak{sl}(E))$  be the canonical filtration of  $U(\mathfrak{sl}(E))$ .

**Definition 2.2.** Let  $M(\rho) = U(\mathfrak{sl}(E)) \otimes_{U(\mathfrak{p})} L_\rho$  be the *generalized Verma module* associated to  $\rho$ . Let  $M_l(\rho) = U_l(\mathfrak{sl}(E)) \otimes_{U(\mathfrak{p})} L_\rho$  be the *canonical filtration* of  $M(\rho)$ .

Since  $U_l(\mathfrak{sl}(E)) \otimes_{U(\mathfrak{p})} L_\rho$  is a  $P$ -submodule of  $M(\rho)$  where  $\mathfrak{p} = \mathrm{Lie}(P)$  it follows  $\{M_l(\rho)\}_{l \geq 0}$  is a filtration of  $M(\rho)$  by  $P$ -modules.

Since  $\dim_K(L_\rho) = 1$  we refer to  $M(\rho)$  as a *scalar generalized Verma module*. By definition this construction give all scalar generalized Verma modules for  $\mathrm{SL}(E)$ .

Define the following sub Lie algebra of  $\mathfrak{sl}(E)$ :  $\mathfrak{n}$  is the subalgebra of matrices  $x$  on the form

$$x = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ * & A_2 & \cdots & 0 \\ * & \vdots & \cdots & 0 \\ * & * & \cdots & A_{k+1} \end{pmatrix}$$

where  $A_i$  is a  $d_i \times d_i$ -matrix with zero entries. It follows there is an isomorphism  $\mathfrak{n} \oplus \mathfrak{p} \cong \mathfrak{sl}(E)$  as vector spaces. Let  $\mathfrak{sl}(E) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$  be the standard triangular decomposition of  $\mathfrak{sl}(E)$  as defined in the previous section. It follows there is an inclusion  $\mathfrak{n} \subseteq \mathfrak{n}_-$  of Lie algebras. Let  $R = R(\mathfrak{sl}(E), \mathfrak{h})$  be the roots of  $\mathfrak{sl}(E)$  with respect to  $\mathfrak{h}$ . Let  $B' = \{\alpha_1, \dots, \alpha_m\}$  be a subset of  $R$  such that the set

$$X_{-\alpha_1}, \dots, X_{-\alpha_m}$$

is a basis for  $\mathfrak{n}$ .

Let  $P = (p_1, \dots, p_m)$  and let  $X^P = X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_m}^{p_m}$ . Let  $\alpha_P = -p_1\alpha_1 - \cdots - p_m\alpha_m$ .

**Lemma 2.3.** *The following holds: The set*

$$\{X^P \otimes w : p_1, \dots, p_m \geq 0\}$$

*is a basis for  $M(\rho)$  as  $K$ -vector space. The natural map*

$$\phi : U(\mathfrak{n}) \rightarrow M(\rho)$$

*defined by*

$$\phi(X^P) = X^P \otimes w$$

*is an isomorphism of left  $\mathfrak{n}$ -modules.*

*Proof.* Since  $\mathfrak{n} \oplus \mathfrak{p} = \mathfrak{sl}(E)$  there is by definition an isomorphism

$$U(\mathfrak{sl}(E)) \cong K\{X^P : p_1, \dots, p_m \geq 0\} U(\mathfrak{p})$$

of free right  $U(\mathfrak{p})$ -modules. We get

$$M(\rho) \cong \{X^P : p_1, \dots, p_m \geq 0\} U(\mathfrak{p}) \otimes_{U(\mathfrak{p})} L_\rho \cong$$

$$K\{X^P \otimes w : p_1, \dots, p_m \geq 0\}.$$

The first claim is proved. One checks the map  $\phi$  is a map of left  $\mathfrak{n}$ -modules and the Lemma is proved.  $\square$

Let  $\omega_i = L_1 + \cdots + L_i \in \mathfrak{h}$  for  $i = 1, \dots, n-1$  be the fundamental weights for  $SL(E)$  and let  $\lambda = \sum_{i=1}^k l_i \omega_{n_i}$ . It follows for all  $x \in \mathfrak{h} \subseteq \mathfrak{p}$  that  $\rho(x) = \lambda(x)$ .

Let  $\rho_U : U(\mathfrak{p}) \rightarrow \text{End}(L_\rho)$  be the *associated morphism* of  $\rho$ . Let  $L$  be the following left ideal of  $U(\mathfrak{p})$ :

$$L = U(\mathfrak{p})\{x - \rho(x)\mathbf{1}_{\mathfrak{p}} : x \in \mathfrak{p}\}.$$

**Proposition 2.4.** *Let  $N = \ker(\rho_U)$ . There is an equality of ideals in  $U(\mathfrak{p})$ :  $N = L$ . Hence  $L$  is a two-sided ideal in  $U(\mathfrak{p})$ .*

*Proof.* The proof follows [1], Section 7. There is a short exact sequence of rings

$$0 \rightarrow N \rightarrow U(\mathfrak{p}) \rightarrow K \rightarrow 0.$$

Let  $x_1, x_2, \dots, x_n$  be a basis with  $\rho(x_1) \neq 0$  and  $\rho(x_2) = \dots = \rho(x_n) = 0$ . Assume  $x = x_1^{v_1} x_2^{v_2} \dots x_n^{v_n} \in U(\mathfrak{p})$ . It follows  $\rho_U(x) = 0$  if and only if  $v_2 + \dots + v_n \geq 1$ . We get by the PBW Theorem a direct sum decomposition

$$U(\mathfrak{p}) = \{x_1^{v_1} x_2^{v_2} \dots x_n^{v_n} : v_2 + \dots + v_n \geq 1\} \oplus \{x_1^{v_1} : v_1 \geq 0\}.$$

It follows there is an inclusion of vector spaces

$$\{x_1^{v_1} x_2^{v_2} \dots x_n^{v_n} : v_2 + \dots + v_n \geq 1\} \subseteq N.$$

One checks there is an equality of vector spaces

$$\{x_1^{v_1} : v_1 \geq 0\} = \{x_1^l \rho(x_1)^l : l \geq 0\}.$$

There is an inclusion

$$\{x_1^{v_1} x_2^{v_2} \dots x_n^{v_n} : v_2 + \dots + v_n \geq 1\} \oplus \{x_1^{v_1} : v_1 \geq 1\} \subseteq L$$

hence  $\text{codim}(L, U(\mathfrak{p})) \leq 1$ . Similarly  $\text{codim}(N, U(\mathfrak{p})) = 1$ . Since  $L \subseteq N$  it follows there is an equality  $L = N$  and the Proposition is proved.  $\square$

**Definition 2.5.** Let  $\text{char}(\rho) = U(\mathfrak{g})\{x - \rho(x)\mathbf{1}_{\mathfrak{g}} : x \in \mathfrak{p}\}$  be the left *character ideal* of  $\rho$  in  $U(\mathfrak{g})$ . Let  $\text{char}_l(\rho) = \text{char}(\rho) \cap U_l(\mathfrak{g})$  be the canonical filtration of  $\text{char}(\rho)$ .

**Proposition 2.6.** The natural map  $\phi : U(\mathfrak{g}) \rightarrow M(\rho)$  defined by  $\phi(x) = x \otimes w$  defines an exact sequence of left  $U(\mathfrak{g})$ -modules

$$0 \rightarrow \text{char}(\rho) \rightarrow U(\mathfrak{g}) \xrightarrow{\phi} M(\rho) \rightarrow 0.$$

*Proof.* Let  $X_{-\alpha_1}, \dots, X_{-\alpha_m}$  be the basis for  $\mathfrak{n}$  constructed above and let

$$X^P = X_{-\alpha_1}^{p_1} \dots X_{-\alpha_m}^{p_m}$$

with  $p_i \geq 0$  integers. There is an isomorphism of right  $U(\mathfrak{p})$ -modules

$$U(\mathfrak{g}) \cong \{X^P : p_i \geq 0\} U(\mathfrak{p})$$

hence we get an isomorphism of vector spaces

$$M(\rho) \cong \{X^P \otimes w : p_i \geq 0\}.$$

Assume  $X \in U(\mathfrak{g})$  is an element. We may write  $X = \sum_P X^P x_P$  with  $x_P \in U(\mathfrak{p})$ . Assume  $\phi(X) = X \otimes w = 0$ . We get

$$\phi(X) = X \otimes w = \sum_P X^P x_P \otimes w = \sum_P X^P \otimes x_P w = 0$$

It follows

$$x_P w = 0$$

for all  $P$  hence  $x_P \in \ker(\rho_U) = U(\mathfrak{p})\{y - \rho(y)\mathbf{1}_{\mathfrak{p}} : y \in \mathfrak{p}\}$ . It follows  $X = \sum_P X^P x_P \in \text{char}(\rho)$  hence  $\ker(\phi) \subseteq \text{char}(\rho)$ . One checks  $\text{char}(\rho) \subseteq \ker(\phi)$  and the Proposition is proved.  $\square$

By Lemma 2.3 it follows

$$M(\rho) = K\{X^P \otimes w : p_i \geq 0\}.$$

Let

$$\alpha_P = -p_1 \alpha_1 - \dots - p_m \alpha_m \in \mathfrak{h}^*.$$

Assume  $x \in \mathfrak{h}$ . We get

$$\begin{aligned} x(X^P \otimes w) &= [x, X^P] \otimes w + X^P x \otimes w = \\ &\alpha_P(x) X^P \otimes w + X^P \otimes xw = \end{aligned}$$

$$\begin{aligned} \alpha_P(x)X^P \otimes w + \lambda(x)X^P \otimes w = \\ (\lambda + \alpha_P)(x)X^P \otimes w. \end{aligned}$$

Hence

$$M(\rho)_{\lambda+\alpha_P} = K\{X^P \otimes w\}.$$

It follows

$$M(\rho) = \oplus_P M(\rho)_{\lambda+\alpha_P}.$$

Define

$$M(\rho)_+ = \oplus_{p_i \geq 1} M(\rho)_{\lambda+\alpha_P}.$$

It follows

$$M(\rho) = M(\rho)_\lambda \oplus M(\rho)_+$$

where

$$M(\rho)_\lambda = K\{\mathbf{1}_{\mathfrak{g}} \otimes w\}.$$

Let the vector  $\mathbf{1}_{\mathfrak{g}} \otimes v \in M(\rho)$  be the *canonical generator* of  $M(\rho)$ .

**Theorem 2.7.** *Let  $M(\rho)$  be the scalar generalized Verma module associated to the character  $\rho$ . The following holds:  $M(\rho)$  contains a maximal non-trivial sub- $\mathfrak{g}$ -module  $K$ . The quotient  $L(\rho) = M(\rho)/K$  is simple and  $\dim_K(L(\rho)) \geq 2$ . Let  $v = \overline{\mathbf{1} \otimes w} \in L(\rho)$ . The ideal  $\text{ann}(v)$  is the largest non-trivial left ideal in  $U(\mathfrak{g})$  containing  $\text{char}(\rho)$ . The vector  $v$  satisfies the following:  $U(\mathfrak{n}_+)v = 0$ . For all  $x \in \mathfrak{h}$  it follows  $xv = \lambda(x)v$  hence  $v$  has weight  $\lambda$*

*Proof.* Consider the element  $X_{-\alpha_m} \otimes w \in M(\rho)_+$  and look at the product

$$X_{\alpha_m} X_{-\alpha_m} \otimes w.$$

We get

$$\begin{aligned} X_{\alpha_m} X_{-\alpha_m} \otimes w &= [X_{\alpha_m}, X_{-\alpha_m}] \otimes w + X_{-\alpha_m} \otimes X_{\alpha_m} w = \\ H_{\alpha_m} \otimes w &= \mathbf{1}_{\mathfrak{g}} \otimes H_{\alpha_m} w = \\ \lambda(H_{\alpha_m})(\mathbf{1}_{\mathfrak{g}} \otimes w). \end{aligned}$$

It follows

$$X_{\alpha_m} X_{-\alpha_m} \otimes w \in M(\rho)_\lambda$$

hence  $M(\rho)_+$  is not  $\mathfrak{g}$ -stable.

Assume  $L \subsetneq M(\rho)$  is a non-trivial  $\mathfrak{g}$ -stable module. It follows  $L \cap M(\rho)_\lambda = 0$  hence  $L \subseteq M(\rho)_+$ . Let  $K$  be the sum of all non-trivial sub- $\mathfrak{g}$ -modules of  $M(\rho)$ . It follows  $K \subseteq M(\rho)_+$  since  $M(\rho)_+$  is not  $\mathfrak{g}$ -stable. Hence  $K \subsetneq M(\rho)$  is a maximal non-trivial sub- $\mathfrak{g}$ -module of  $M(\rho)$  and  $L(\rho) = M(\rho)/K$  is a simple quotient. It follows  $\dim_K(L(\rho)) \geq 2$ .

One checks the vector  $v$  is annihilated by  $U(\mathfrak{n}_+)$  and has weight  $\lambda$ .

By Proposition 2.6 there is an exact sequence of left  $U(\mathfrak{g})$ -modules

$$0 \rightarrow \text{char}(\rho) \rightarrow U(\mathfrak{g}) \rightarrow M(\rho) \rightarrow 0$$

hence there is an isomorphism

$$M(\rho) \cong U(\mathfrak{g})/\text{char}(\rho)$$

of left  $U(\mathfrak{g})$ -modules. It follows there is a bijection between the set of left sub- $\mathfrak{g}$ -modules of  $M(\rho)$  and left sub- $\mathfrak{g}$ -modules of  $U(\mathfrak{g})/\text{char}(\rho)$ . This induce a bijection between the set of left ideals in  $U(\mathfrak{g})$  containing the ideal  $\text{char}(\rho)$  and the set of left sub- $\mathfrak{g}$ -modules of  $M(\rho)$ . It follows the submodule  $K$  corresponds to a maximal

non-trivial left ideal  $J$  in  $U(\mathfrak{g})$  containing  $\text{char}(\rho)$ . The ideal  $J$  is by definition the annihilator ideal of  $v$ : There is an equality

$$J = \text{ann}(v).$$

The Theorem is proved.  $\square$

**Corollary 2.8.** *The following holds:*

$$(2.8.1) \quad L(\rho) \text{ is simple for all } \underline{l} \in \mathbf{Z}^k.$$

$$(2.8.2) \quad \text{If } \underline{l} \neq \underline{l}' \text{ it follows } L(\rho_{\underline{l}}) \neq L(\rho_{\underline{l}'}).$$

$$(2.8.3) \quad \text{If } l_1, \dots, l_k < 0. \text{ It follows } \dim_K(L(\rho)) = \infty$$

*Proof.* Claim 2.8.1: This is by definition of  $L(\rho)$  since the submodule  $K$  is maximal. Claim 2.8.2: Assume  $\phi : L(\rho_{\underline{l}}) \rightarrow L(\rho_{\underline{l}'})$  is a map of  $\mathfrak{g}$ -modules. It follows  $\phi$  is the zero map or an isomorphism since the modules are simple. If it is an isomorphism it follows the weights are equal. This implies  $\underline{l} = \underline{l}'$  a contradiction. The claim is proved. We prove claim 2.8.3: Assume  $\dim_K(L(\rho)) < \infty$ . It follows  $L(\rho) \cong V_\lambda$  where  $V_\lambda$  has a highest weight vector  $v'$  with highest weight  $\lambda' = \sum_{j=1}^{n-1} l'_j \omega_j$  with  $l'_j \geq 0$ . Since the vector  $v$  in  $L(\rho)$  has weight  $\lambda$  with  $l_i < 0$  one gets a contradiction. The Corollary follows.  $\square$

Assume  $\mathfrak{p} = \mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ . It follows from [1], Section 7 the  $\text{SL}(E)$ -module  $L(\lambda + \delta)$  is isomorphic to  $V_\lambda$  - the finite dimensional irreducible  $\text{SL}(E)$ -module with highest weight  $\lambda = \sum_{i=1}^k l_i \omega_{n_i}$ . Here  $l_i \geq 1$  is an integer for  $i = 1, \dots, k$ . Hence the class

$$\{L(\rho_{\underline{l}}) : \rho_{\underline{l}} : \mathfrak{p} \rightarrow K, \underline{l} \in \mathbf{Z}^k\}$$

is a class of mutually non-isomorphic  $\text{SL}(E)$ -modules parametrized by a parabolic subalgebra  $\mathfrak{p} \subseteq \mathfrak{sl}(E)$  and a character  $\rho : \mathfrak{p} \rightarrow K$  containing the class of all finite dimensional irreducible  $\text{SL}(E)$ -modules.

Let  $K_l = K \cap M_l(\rho) \subseteq M(\rho)$  be the induced filtration on  $K$ . There is an exact sequence of  $P$ -modules

$$0 \rightarrow K_l \rightarrow M_l(\rho) \rightarrow L_l(\rho) \rightarrow 0.$$

**Definition 2.9.** Let  $\{L_l(\rho)\}_{l \geq 0}$  be the *canonical filtration* of  $L(\rho)$ .

It follows  $\{L_l(\rho)\}_{l \geq 0}$  is a filtration of  $L(\rho)$  by  $P$ -modules.

**Corollary 2.10.** *Assume  $L(\rho) = V_\lambda$  is a finite dimensional irreducible  $\text{SL}(E)$ -module with highest weight vector  $v$  and highest weight  $\lambda$ . It follows  $L_l(\rho) \cong U_l(\mathfrak{sl}(E))v$ .*

*Proof.* The proof is obvious.  $\square$

### 3. CLASSICAL VERMA MODULES AND ANNIHILATOR IDEALS

Let  $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ . It follows  $\mathfrak{b}_+$  is a sub Lie algebra of  $\mathfrak{sl}(E)$ . Let  $(x, n), (y, m)$  be elements of  $\mathfrak{b}_+$ . The Lie product on  $\mathfrak{sl}(E)$  induce the following product on  $\mathfrak{b}_+$ : define the following action of  $\mathfrak{h}$  on  $\mathfrak{n}_+$ .

$$\text{ad} : \mathfrak{h} \rightarrow \text{End}(\mathfrak{n}_+)$$

$$\text{ad}(x)(n) = [x, n].$$

It follows  $\mathfrak{n}_+$  is a  $\mathfrak{h}$ -module. Let  $\text{ad}(x)(h) = x(h)$ .

**Lemma 3.1.** *Define*

$$[(x, n), (y, m)] = (0, x(m) - y(n) + [n, m])$$

where  $[\cdot, \cdot]$  is the bracket on  $\mathfrak{n}_+$ . It follows the natural injection  $\mathfrak{b}_+ \rightarrow \mathfrak{sl}(E)$  is a map of Lie algebras.

*Proof.* The proof is obvious.  $\square$

Let  $\mu \in \mathfrak{h}^*$  be a linear form on  $\mathfrak{h}$ . Let  $L_\mu = Kw$  be the free rank one  $K$ -module on  $w$ . Define the following map

$$\tau_\mu : \mathfrak{b}_+ \rightarrow \text{End}(L_\mu)$$

by

$$\tau_\mu(x, n)(v) = \mu(x)v.$$

**Lemma 3.2.** *The map  $\tau_\mu$  makes  $L_\mu$  into a  $\mathfrak{b}_+$ -module.*

*Proof.* It is clear  $[\tau_\mu(x, n), \tau_\mu(y, m)] = \tau_\mu([(x, n), (y, m)])$  hence the claim is proved.  $\square$

It follows  $L_\mu$  is a left  $\mathfrak{b}_+$ -module. By definition  $U(\mathfrak{sl}(E))$  is a right  $\mathfrak{b}_+$ -module and we may form the tensor product

$$M(\mu) = U(\mathfrak{sl}(E)) \otimes_{U(\mathfrak{b}_+)} L_\mu.$$

It follows  $M(\mu)$  is a left  $G$ -module.

**Definition 3.3.** The  $G$ -module  $M(\mu)$  is the *Verma module* associated to the linear form  $\mu \in \mathfrak{h}^*$ .

Let  $\delta = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$  and let  $\lambda$  be an element in  $\mathfrak{h}$ . Let  $L_{\lambda-\delta} = Kv$  be the free rank one  $K$ -vectorspace on the element  $v$ . By the result above we get a character

$$\tau_{\lambda-\delta} : \mathfrak{b}_+ \rightarrow \text{End}(L_{\lambda-\delta})$$

defined by

$$\tau_{\lambda-\delta}(h, n) = (\lambda - \delta)(h)$$

where  $(h, n)$  is an element in  $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ . Let  $\alpha_1, \dots, \alpha_m$  be the  $m$  distinct elements of  $R_+$ . Let  $X_{-\alpha_i}$  be an element in  $\mathfrak{g}^{-\alpha_i}$ . It follows the set

$$X_{-\alpha_1}, \dots, X_{-\alpha_m}$$

is a basis for  $\mathfrak{n}_-$  as vector space. There is a decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}_+$  and it follows  $U(\mathfrak{g})$  is a free right  $U(\mathfrak{b}_+)$ -module as follows:

$$U(\mathfrak{g}) = \{X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_m}^{p_m} : p_1, \dots, p_m \geq 0\} U(\mathfrak{b}_+).$$

**Lemma 3.4.** *Assume  $X^P = X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_m}^{p_m}$  with  $p_1, \dots, p_m \geq 0$  integers. Let  $x \in \mathfrak{h}$ . It follows*

$$x(X^P) = (-p_1\alpha_1 - \cdots - p_m\alpha_m)(x)X^P = \alpha_P(x)X^P.$$

*Proof.* The proof is an easy calculation.  $\square$

Let  $P = (p_1, \dots, p_m)$  with  $p_i$  integers and let  $\alpha_P = p_1\alpha_1 + \cdots + p_m\alpha_m$ . Let  $X^P = X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_m}^{p_m}$ .

**Proposition 3.5.** *Assume  $x \in \mathfrak{h}$ . The following holds:*

$$(3.5.1) \quad M(\lambda) = K\{X^P \otimes v : p_1, \dots, p_m \geq 0\}$$

$$(3.5.2) \quad x(X^P \otimes v) = (\lambda - \delta - \alpha_P)(x)X^P \otimes v$$

$$(3.5.3) \quad M(\lambda) = \bigoplus_{p_1, \dots, p_m \geq 0} M(\lambda)_{\lambda - \delta - \alpha_P}$$

$$(3.5.4) \quad M(\lambda)_{\lambda - \delta} = K(1 \otimes v)$$

$$(3.5.5) \quad M(\lambda)_\mu = K\{X^P \otimes v : \lambda - \delta - \alpha_P = \mu\}$$

$$(3.5.6) \quad U(\mathfrak{n}_-)(1 \otimes v) = M(\lambda)$$

*Proof.* We prove Claim 3.5.1: There is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{b}_+$$

and the set

$$X_{-\alpha_1}, \dots, X_{-\alpha_m}$$

is a basis for  $\mathfrak{n}_-$  as  $K$ -vector space. It follows by the PBW-theorem that the set

$$\{X^P : p_1, \dots, p_m \geq 0\}$$

is a basis for  $U(\mathfrak{n}_-)$  as  $K$ -vector space. It follows  $U(\mathfrak{g})$  is isomorphic to

$$\{X^P : p_1, \dots, p_m \geq 0\} U(\mathfrak{b}_+)$$

as free left  $U(\mathfrak{b}_+)$ -module. We get

$$\begin{aligned} M(\lambda) &= U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} L_{\lambda - \delta} \cong \\ &\{X^P : p_1, \dots, p_m \geq 0\} U(\mathfrak{b}_+) \otimes_{U(\mathfrak{b}_+)} L_{\lambda - \delta} \cong \\ &K\{X^P \otimes v : p_1, \dots, p_m \geq 0\} \end{aligned}$$

and Claim 3.5.1 is proved.

We prove Claim 3.5.2: By the previous Lemma it follows for all  $x \in \mathfrak{h}$

$$[x, X^P] = \alpha_P(x)X^P.$$

We get

$$\begin{aligned} x(X^P \otimes v) &= [x, X^P] \otimes v + X^P x \otimes v = \\ &\alpha_P(x)X^P \otimes v + X^P \otimes (\lambda - \delta)(x)v = \\ &(\lambda - \delta - \alpha_P)(x)X^P \otimes v \end{aligned}$$

and Claim 3.5.2 is proved. We prove Claim 3.5.3: By Claim 3.5.1 it follows  $M(\lambda)$  has a basis given as follows:

$$\{X^P \otimes v : p_1, \dots, p_m \geq 0\}.$$

Since for all  $x \in \mathfrak{h}$  it follows

$$x(X^P \otimes v) = (\lambda - \delta - \alpha_P)(x)X^P \otimes v$$

Claim 3.5.3 follows.

We prove Claim 3.5.4: Let  $X^P \otimes v \in M(\lambda)$  It follows  $x(X^P \otimes v) = (\lambda - \delta - \alpha_P)(x)X^P \otimes v$ . Hence  $X^P \otimes v$  is in  $M(\lambda)_{\lambda - \delta}$  if and only if

$$\lambda - \delta - \alpha_P = \lambda - \delta.$$

Since  $p_1, \dots, p_m \geq 0$  this can only occur when  $p_1 = \dots = p_m = 0$  hence  $\alpha_P = 0$ . The Claim is proved.

Claim 3.5.5 is by definition.



We prove Claim 3.5.6: Since the map

$$\phi : \mathbf{U}(\mathfrak{n}_-) \rightarrow M(\lambda)$$

defined by

$$\phi(X^P) = X^P \otimes v$$

is an isomorphism of  $\mathfrak{n}_-$ -modules Claim 3.5.6 follows. The Proposition is proved.  $\square$

**Definition 3.6.** The element  $1 \otimes v \in M(\lambda)$  is called the *canonical generator* of  $M(\lambda)$ .

**Proposition 3.7.** *There is an isomorphism of  $\mathfrak{n}_-$ -modules*

$$\phi : \mathbf{U}(\mathfrak{n}_-) \cong M(\lambda)$$

defined by

$$\phi(X^P) = X^P \otimes v.$$

*Proof.* Let  $\mathfrak{n}_-$  have basis  $X_{-\alpha_1}, \dots, X_{-\alpha_m}$  and let  $X^P = X_{-\alpha_1}^{p_1} \cdots X_{-\alpha_m}^{p_m}$ . It follows  $M(\lambda)$  has a basis consisting of elements  $X^P \otimes v$ . Hence the map  $\phi$  is an isomorphism of vector spaces. One checks it is  $\mathfrak{n}_-$ -linear and the Proposition follows.  $\square$

Let in the following  $E = K\{e_1, \dots, e_n\}$  be an  $n$ -dimensional vector space over  $K$  and let  $\mathfrak{g} = \mathfrak{sl}(E)$  with  $\mathfrak{h}$  in  $\mathfrak{g}$  the abelian sub-algebra of diagonal matrices. Let  $E_i = K\{e_1, \dots, e_i\}$  for  $i = 1, \dots, n-1$ . It follows we get a complete flag

$$E_\bullet : 0 \neq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_{n-1} \subseteq E$$

in  $E$  with  $\dim(E_i) = i$ . Let  $\underline{l} = (l_1, \dots, l_{n-1})$  with  $l_i \geq 0$  integers. Let

$$W(\underline{l}) = \text{Sym}^{l_1}(E) \otimes \text{Sym}^{l_2}(\wedge^2 E) \otimes \cdots \otimes \text{Sym}^{l_{n-1}}(\wedge^{n-1} E).$$

Let  $v_i = \wedge^i E_i$  and let  $v = v_1 l_1 \otimes \cdots \otimes v_{n-1}^{l_{n-1}}$  be a line in  $W(\underline{l})$ . Let  $P$  in  $SL(E)$  be the subgroup of elements fixing the flag  $E_\bullet$ . It follows  $P$  consists of upper triangular matrices with determinant one. Let  $\mathfrak{p} = \text{Lie}(P)$ . It follows  $\mathfrak{p}$  consists of upper triangular matrices with trace zero. Let

$$\omega_i = L_1 + \cdots + L_i$$

where

$$\mathfrak{h}^* = K\{L_1, \dots, L_n\}/L_1 + \cdots + L_n.$$

It follows  $\omega_1, \dots, \omega_{n-1}$  are the fundamental weights for  $\mathfrak{g}$ . Let  $\lambda = \sum_{i=1}^{n-1} l_i \omega_i$ .

**Proposition 3.8.** *Let  $x \in \mathfrak{p}$  be an element. The following formula holds:*

$$x(v) = \sum_{i=1}^{n-1} l_i (a_{11} + \cdots + a_{ii})v$$

where  $a_{ii}$  is the  $i$ 'th diagonal element of  $x$ . It follows the vector  $v$  has weight  $\lambda$ . The  $SL(E)$ -module  $V_\lambda$  generated by  $v$  is an irreducible finite dimensional  $SL(E)$ -module with highest weight vector  $v$  and highest weight  $\lambda$ .

*Proof.* The Proposition follows from [1], Section 7 and an explicit calculation.  $\square$

Let  $L_v$  in  $V_\lambda$  be the line spanned by the vector  $v$ . It follows the subgroup of  $\text{SL}(E)$  of elements fixing  $L_v$  equals the group  $P$ . We get a character

$$\rho : \mathfrak{p} \rightarrow \text{End}(L_v)$$

defined by

$$\rho(x)v = x(v) = \sum_{i=1}^{n-1} l_i(a_{11} + \cdots + a_{ii})v.$$

We get an exact sequence of Lie algebras

$$0 \rightarrow \mathfrak{p}_v \rightarrow \mathfrak{p} \rightarrow \text{End}(L_v) \rightarrow 0$$

where  $\mathfrak{p}_v = \ker(\rho)$ .

**Definition 3.9.** Let

$$\text{char}(\rho) = \text{U}(\mathfrak{g})\{x - \rho(x)\mathbf{1}_{\mathfrak{g}} : x \in \mathfrak{p}\}$$

be the left *character ideal* of  $\rho$ . Let

$$\text{ann}(v) = \{x \in \text{U}(\mathfrak{g}) : x(v) = 0\}$$

be the left *annihilator ideal* of  $v$ . Let

$$\text{char}_l(\rho) = \text{char}(\rho) \cap \text{U}_l(\mathfrak{g})$$

and

$$\text{ann}_l(v) = \text{ann}(v) \cap \text{U}_l(\mathfrak{g})$$

for all  $l \geq 1$ .

If  $x \in \mathfrak{h}$  it follows  $\rho(x) = \lambda(x)$  where  $\lambda$  is the highest weight of  $V_\lambda$ . By [1], Section 7 the following holds: Let for  $\alpha \in B$  the integer  $m_\alpha$  be defined as follows:

$$m_\alpha = \lambda(H_\alpha) + 1$$

where

$$0 \neq H_\alpha \in [X_\alpha, X_{-\alpha}].$$

If  $\alpha_i = L_i - L_{i+1}$  it follows  $X_{\alpha_i} = E_{i,i+1}$ . It also follows  $X_{-\alpha_i} = E_{i+1,i}$ . We get

$$H_{\alpha_i} = E_{ii} - E_{i+1,i+1}.$$

**Lemma 3.10.** *The following holds for all  $i = 1, \dots, n-1$ :*

$$m_{\alpha_i} = l_i + 1.$$

*Proof.* The proof is an easy calculation. □

We get by [1], Proposition 7.2.7, Section 7 the following description of the ideal  $\text{ann}(v)$  in  $\text{U}(\mathfrak{g})$ . Note that  $\text{U}(\mathfrak{g})$  is a noetherian associative algebra and the ideal  $\text{ann}(v)$  is a left ideal in  $\text{U}(\mathfrak{g})$ . It follows  $\text{ann}(v)$  has a finite set of generators. The following holds: Let

$$I(v) = \text{U}(\mathfrak{g})\mathfrak{n}_+ + \sum_{x \in \mathfrak{h}} \text{U}(\mathfrak{g})(x - \lambda(x)\mathbf{1}_{\mathfrak{g}}).$$

It follows  $I(v) \subseteq \text{U}(\mathfrak{g})$  is a left ideal. It follows

$$\text{ann}(v) = I(v) + \sum_{\alpha \in B} \text{U}(\mathfrak{g})X_{-\alpha}^{m_\alpha}.$$

Let  $I_l(v) = I(v) \cap U_l(\mathfrak{g})$  for all  $l \geq 1$  integers. Let for any element  $x \in U(\mathfrak{g})$

$$fil(x) = \min\{l : x \in U_l(\mathfrak{g})\}$$

be the *filtration* of the element  $x$ .

**Lemma 3.11.** *There is for every integer  $l \geq 1$  an equality*

$$ann_l(v) = I_l(v) + \sum_{\alpha \in B} U_{l-m_\alpha}(\mathfrak{g}) X_{-\alpha}^{m_\alpha}$$

*of vector spaces.*

*Proof.* The inclusion

$$I_l(v) + \sum_{\alpha \in B} U_{l-m_\alpha}(\mathfrak{g}) X_{-\alpha}^{m_\alpha} \subseteq ann_l(v)$$

is obvious. Assume

$$x = u + v \in ann_l(v) = (I(v) + \sum_{\alpha \in B} U(\mathfrak{g}) X_{-\alpha}^{m_\alpha}) \cap U_l(\mathfrak{g})$$

with

$$u \in I(v)$$

and

$$v \in \sum_{\alpha \in B} U(\mathfrak{g}) X_{-\alpha}^{m_\alpha}.$$

It follows  $fil(u), fil(v) \leq fil(x)$ . Hence  $u, v \in U_l(\mathfrak{g})$ . It follows  $u \in I_l(v)$  and

$$v \in \sum_{\alpha \in B} U_{l-m_\alpha}(\mathfrak{g}) X_{-\alpha}^{m_\alpha}.$$

The Lemma follows. □

**Proposition 3.12.** *There is an equality*

$$I(v) = char(\rho)$$

*of left ideals in  $U(\mathfrak{g})$ .*

*Proof.* Recall there is a triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ . It follows there is an equality  $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{b}_+$  of Lie algebras. Assume  $x \in \mathfrak{n}_+ \subseteq \mathfrak{p}$ . It follows  $\rho(x) = 0$ , hence  $x(v) = \rho(x)v = 0$ . Assume  $x \in \mathfrak{h} \subseteq \mathfrak{p}$ . It follows  $\rho(x) = \lambda(x)$  where  $\lambda$  is the highest weight of  $V_\lambda$  with

$$\lambda = \sum_{i=1}^{n-1} l_i \omega_i.$$

It follows there is an element  $x \in \mathfrak{h}$  with  $\lambda(x) \neq 0$ . Moreover we may choose elements  $h_1, \dots, h_{n-2}$  in  $\mathfrak{h}$  with  $\rho(h_i) = \lambda(h_i) = 0$  and

$$\{x, h_1, \dots, h_{n-2}\}$$

is a basis for  $\mathfrak{h}$ . Assume  $y(x - \rho(x)\mathbf{1}_{\mathfrak{g}}) \in char(\rho)$ . It follows

$$x \in \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{n}_+.$$

It follows  $x = x_1 + x_2$  with  $x_1 \in \mathfrak{h}$  and  $x_2 \in \mathfrak{n}_+$ . It follows

$$y(x - \rho(x)\mathbf{1}_{\mathfrak{g}}) = y(x_1 + x_2 - \rho(x_1 + x_2)\mathbf{1}_{\mathfrak{g}}) = y(x_1 - \rho(x_1)\mathbf{1}_{\mathfrak{g}}) + y(x_2 - \rho(x_2)\mathbf{1}_{\mathfrak{g}}).$$

Since  $x_1 \in \mathfrak{h}$  it follows

$$y(x_1 - \rho(x_1)\mathbf{1}_{\mathfrak{g}}) = y(x_1 - \lambda(x_1)\mathbf{1}_{\mathfrak{g}}) \in \sum_{x \in \mathfrak{h}} \mathbf{U}(\mathfrak{g})(x - \lambda(x)\mathbf{1}_{\mathfrak{g}}) \subseteq I(v).$$

Since  $x_2 \in \mathfrak{n}_+$  it follows  $\rho(x_2) = 0$ . We get

$$y(x_2 - \rho(x_2)\mathbf{1}_{\mathfrak{g}}) = yx_2 \in \mathbf{U}(\mathfrak{g})\mathfrak{n}_+.$$

It follows  $y(x - \rho(x)\mathbf{1}_{\mathfrak{g}}) \in I(v)$  and we have proved the inclusion  $\text{char}(\rho) \subseteq I(v)$ . We prove the reverse inclusion: Assume  $x = x_1 + x_2 \in I(v)$  with  $x_1 = yn \in \mathbf{U}(\mathfrak{g})\mathfrak{n}_+$  and  $y \in \mathbf{U}(\mathfrak{g})$  and  $n \in \mathfrak{n}_+$ . Assume also  $x_2 = z(u - \lambda(u)\mathbf{1}_{\mathfrak{g}})$  with  $z \in \mathbf{U}(\mathfrak{g})$  and  $u \in \mathfrak{h}$ . It follows  $\rho(n) = 0$  and  $x_1 = y(n - \rho(n)\mathbf{1}_{\mathfrak{g}}) \in \text{char}(\rho)$ . Also  $x_2 = z(u - \lambda(u)\mathbf{1}_{\mathfrak{g}}) = z(u - \rho(u)\mathbf{1}_{\mathfrak{g}}) \in \text{char}(\rho)$ . It follows  $x = x_1 + x_2 \in \text{char}(\rho)$  and we have proved the reverse inclusion  $I(v) \subseteq \text{char}(\rho)$ . The Proposition is proved.  $\square$

**Corollary 3.13.** *There is for every  $l \geq 1$  an equality*

$$I_l(v) = \text{char}_l(\rho)$$

*of filtrations.*

*Proof.* The Corollary follows from Proposition 3.12.  $\square$

Recall  $\lambda = \sum_{i=1}^{n-1} l_i \omega_i$ . Define  $m(\lambda) = \min_{i=1, \dots, n-1} \{l_i\}$ .

**Theorem 3.14.** *For all  $1 \leq l \leq m(\lambda)$  there is an equality*

$$\text{ann}_l(v) = \text{char}_l(\rho)$$

*of filtrations.*

*Proof.* By Lemma 3.11 there is an equality

$$\text{ann}_l(v) = I_l(v) + \sum_{\alpha_i \in B} \mathbf{U}_{l-m_{\alpha_i}}(\mathfrak{g}) X_{-\alpha_i}^{m_{\alpha_i}}$$

of vector spaces. We get

$$X_{-\alpha_i}^{m_{\alpha_i}} = E_{i+1,i}^{l_i+1}$$

for  $i = 1, \dots, n-1$ . We get from Corollary 3.13

$$\begin{aligned} \text{ann}_l(v) &= I_l(v) + \sum_{i=1}^{n-1} \mathbf{U}_{l-l_i-1}(\mathfrak{g}) E_{i+1,i}^{l_i+1} = \\ &= \text{char}_l(\rho) + \sum_{i=1}^{n-1} \mathbf{U}_{l-l_i-1}(\mathfrak{g}) E_{i+1,i}^{l_i+1}. \end{aligned}$$

If  $1 \leq l \leq m(\lambda)$  it follows  $l \leq l_i$  for all  $i = 1, \dots, n-1$ . It follows  $l - l_i - 1 < 0$  hence we get an equality

$$\text{ann}_l(v) = \text{char}_l(\rho)$$

and the claim of the Theorem follows.  $\square$

Let  $\mathfrak{p} \subseteq \mathfrak{sl}(E)$  be a parabolic subalgebra and let  $\mathfrak{n}$  be the complementary Lie algebra as defined in [3]. It follows there is a direct sum decomposition as vector spaces  $\mathfrak{p} \oplus \mathfrak{n} \cong \mathfrak{sl}(E)$ . We get inclusions  $\mathfrak{n} \subseteq \mathfrak{n}_-$  and  $\mathfrak{b}_+ \subseteq \mathfrak{p}$  of Lie algebras. Let  $\rho : \mathfrak{p} \rightarrow \text{End}(L_\rho)$  with induced character

$$\tilde{\rho} : \mathfrak{b}_+ \rightarrow \text{End}(L_{\tilde{\rho}}).$$

**Lemma 3.15.** *There is a surjective map of left  $U(\mathfrak{sl}(E))$ -modules*

$$\phi : M(\tilde{\rho}) \rightarrow M(\rho).$$

*It follows  $\ker(\phi) = \text{char}(\rho)/\text{char}(\tilde{\rho})$ .*

*Proof.* There is a natural map

$$f : U(\mathfrak{sl}(E)) \times L_\rho \rightarrow U(\mathfrak{sl}(E)) \otimes_{U(\mathfrak{p})} L_\rho$$

defined by

$$f(z, w) = z \otimes w.$$

This map induce a surjection

$$\phi : U(\mathfrak{sl}(E)) \otimes_{U(\mathfrak{b}_+)} L_\rho \rightarrow U(\mathfrak{sl}(E)) \otimes_{U(\mathfrak{p})} L_\rho$$

of left  $U(\mathfrak{sl}(E))$ -modules and the first claim is proved. The second claim follows easily and the Lemma is proved.  $\square$

Let  $K \subseteq M(\rho)$  be the unique maximal  $\mathfrak{sl}(E)$ -stable submodule and let  $K' \subseteq M(\tilde{\rho})$  be the unique  $\mathfrak{sl}(E)$ -stable submodule. It follows  $\phi(K') \subseteq K$  since the image of a  $\mathfrak{sl}(E)$ -stable module is  $\mathfrak{sl}(E)$ -stable. We get a commutative diagram of maps of  $\mathfrak{sl}(E)$ -modules where the middle vertical arrow is surjective:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K' & \longrightarrow & M(\tilde{\rho}) & \xrightarrow{p'} & L(\tilde{\rho}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \phi & & \downarrow \phi' \\ 0 & \longrightarrow & K & \longrightarrow & M(\rho) & \xrightarrow{p} & L(\rho) \longrightarrow 0 \end{array}.$$

**Lemma 3.16.** *The map  $\phi'$  is an isomorphism of  $\mathfrak{sl}(E)$ -modules.*

*Proof.* Since the map  $\phi$  is a surjection of  $\mathfrak{sl}(E)$ -modules it follows  $\phi'$  is a surjection. Since the modules  $L(\rho)$  and  $L(\tilde{\rho})$  are simple it follows  $\ker(\phi')$  is zero. The Lemma is proved.  $\square$

Hence we get no new simple  $SL(E)$ -modules when we consider simple quotients  $L(\rho)$  where  $\mathfrak{p}$  is a parabolic subalgebra of  $\mathfrak{sl}(E)$  with  $\mathfrak{p} \neq \mathfrak{b}_+$ . The modules  $L(\rho)$  may be constructed using simple quotients of classical Verma modules as done in [1].

Let  $u = 1 \otimes w \in M(\rho)$  be the canonical generator and let  $v$  be its image in  $L(\rho)$  under the canonical projection map  $M(\rho) \rightarrow L(\rho)$ . Let  $\text{ann}(v)$  be the left annihilator ideal in  $U(\mathfrak{sl}(E))$  of the vector  $v$ .

Recall the exact sequence

$$0 \rightarrow K_l \rightarrow M_l(\rho) \rightarrow L_l(\rho) \rightarrow 0$$

from the previous section.

**Proposition 3.17.** *The following holds:*

$$K_l = 0 \text{ if and only if } \text{ann}_l(v) = \text{char}_l(\rho).$$

*Proof.* There is a commutative diagram of exact sequences of left  $\mathfrak{sl}(E)$ -modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{char}(\rho) & \longrightarrow & U(\mathfrak{sl}(E)) & \xrightarrow{\phi} & M(\rho) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{ann}(v) & \longrightarrow & U(\mathfrak{sl}(E)) & \xrightarrow{\tilde{\phi}} & L(\rho) & \longrightarrow & 0 \end{array}$$

One checks that  $\phi(\text{ann}(v)) = K$  and  $\phi(\text{ann}_l(v)) = K_l$  for all  $l \geq 1$ . Assume  $\text{ann}_l(v) = \text{char}_l(\rho)$ . It follows

$$K_l = \phi(\text{ann}_l(v)) = \phi(\text{char}_l(\rho)) = 0$$

since the diagram above is exact. Assume  $K_l = 0$  and let  $x \in \text{ann}_l(v)$ . It follows

$$\phi(x) = x \otimes w \in K_l$$

hence  $\phi(x) = 0$ . It follows  $x \in \text{char}_l(\rho)$  since the diagram has exact rows. It follows  $\text{ann}_l(v) = \text{char}_l(\rho)$  and the Proposition follows.  $\square$

Assume  $\mathfrak{p} = \mathfrak{b}_+$  and  $\rho|_{\mathfrak{h}} = \lambda$  with  $\lambda = \sum_i l_i \omega_i$  and  $l_i \geq 0$  for all  $i$ .

**Corollary 3.18.** *If  $1 \leq l \leq m(\lambda)$  it follows there is an isomorphism*

$$M_l(\rho) \cong L_l(\rho)$$

*of left  $\text{SL}(E)$ -modules.*

*Proof.* The Corollary follows from Proposition 3.17 and Theorem 3.14.  $\square$

Let  $P \subseteq \text{SL}(E)$  be the parabolic subgroup with  $\text{Lie}(\mathfrak{p}) = P$  and let  $\mathcal{L} \in \text{Pic}^{\text{SL}(E)}(\text{SL}(E)/P)$  be the linebundle with  $H^0(\text{SL}(E)/P, \mathcal{L})^* = L(\rho) = V_\lambda$ . Let  $X = \text{SL}(E)/P$  and let  $\mathcal{P}_X^l(\mathcal{L})$  be the  $l$ 'th order jetbundle of  $\mathcal{L}$  as defined in [2].

**Corollary 3.19.** *For all  $1 \leq l \leq m(\lambda)$  it follows there is an isomorphism*

$$\mathcal{P}_X^l(\mathcal{L})(e)^* \cong U_l(\mathfrak{sl}(E)) \otimes_{U(\mathfrak{p})} L_\rho$$

*of  $P$ -modules.*

*Proof.* By [2], Theorem 3.10 there is an isomorphism

$$\mathcal{P}_X^l(\mathcal{L})(e)^* \cong U_l(\mathfrak{sl}(E))v \cong L_l(\rho)$$

of  $P$ -modules when  $1 \leq l \leq m(\lambda)$ . The Corollary now follows from Corollary 3.18.  $\square$

Corollary 3.19 gives an algebraic proof of a result proved in [4] over the complex numbers.

## REFERENCES

- [1] J. Dixmier, Enveloping Algebras, Graduate Studies in Mathematics, *American Math. Society* (1996)
- [2] H. Maakestad, On jetbundles and generalized Verma modules II, *Preprint math arXiv 0903.3291*
- [3] H. Maakestad, On the annihilator ideal of a highest weight vector, *Preprint math arXiv:1003.3522*
- [4] W. Smoke, Invariant differential operators, *Trans. Am. Math. Soc.* no. 127 (1967)

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